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# Uniform Asymptotic Solutions of the Reduced Wave Equation

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## 1. INTRODUCTION

Asymptotic solutions of boundary value problems for the reduced wave equation

$$\nabla^2 u + k^2 n^2(\mathbf{x}) u = 0,$$

where $k$ is a large parameter, have been extensively developed and used in recent years. In the regions where geometrical optics is valid (see Lewis and Keller [1] for details and references), the asymptotic solution of (1) in the form $u = e^{ik\varphi} v$, where $\varphi$ and $v$ are functions of the independent variables $\mathbf{x} = (x_1, x_2, x_3)$, and $v$ is a formal series in inverse powers of $k$, has been employed with great success. The geometrical theory of diffraction developed by Keller [2] to extend geometrical optics into regions where it predicts an incomplete asymptotic solution, has increased the range of usefulness of expansions of the above form.

However, in transition regions near caustics, shadow boundaries or source points, where the amplitude terms $v$ become unbounded, modified forms of asymptotic expansions are needed. The boundary layer theory which involves the stretching of coordinates to produce a boundary layer expansion, which must then be matched with outer expansions of the above form, is one approach towards the problem of transition regions [3], [4]. We will be concerned here with a second method, that of uniform asymptotic expansions. It is similar to that developed by Langer, Olver (see [5], [6], respectively), and others, to treat turning point problems for ordinary differential equations. It has been used by Ludwig [7] and by Lewis [8], et al., to obtain asymptotic expansions near caustics and shadow boundaries, respectively. While specific problems have been solved by the uniform expansion method, a general theory, of the type that exists for turning point problems, has not been

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presented for (1). It is our purpose to show how a variety of transition phenomena occurring in the geometrical theory of (1), based on expansions of the form $u = e^{ik\varphi}v$, can be treated in a systematic manner by means of uniform asymptotic expansions. A transition region algorithm will be introduced of which the "classical" algorithm presented above is a special case. By establishing a connection between turning points and transition domains, it will be seen how to choose the appropriate special functions that must be used in the uniform expansions, by obtaining the "related equation" for each of the transition domains to be considered.

## 2. Turning Points and Transition Regions

We recall the classical asymptotic theories for (1) and second order linear ordinary differential equations with a large parameter. The theory for (1) proceeds from the classical algorithm [1]

$$u(\mathbf{x}, k) = e^{ik\varphi(\mathbf{x})}v(\mathbf{x}, k) = e^{ik\varphi} \sum_{n=0}^{\infty} v_n(\mathbf{x})\,(ik)^{-n}. \tag{2}$$

Introducing (2) into (1) and equating different powers of $k$ to zero gives

$$(\nabla\varphi)^2 = n^2(\mathbf{x}), \tag{3}$$

$$2\nabla\varphi \cdot \nabla v_n + (\nabla^2\varphi)\,v_n = -\nabla^2 v_{n-1}; \qquad n = 0, 1, 2, \ldots; \quad v_{-1} = 0. \tag{4}$$

The transport Eqs. (4) are ordinary differential equations along the characteristic curves of the eiconal Eq. (3). In a boundary value or source problem for (1), initial conditions for (3) and (4) are given. Regions where the $v_n$ become unbounded or discontinuous are called transition regions. Other forms of asymptotic expansions for (1) must be sought there.

The classical algorithm for ordinary differential equations

$$y''(x) + k^2 p(x, k)\,y(x) = 0, \tag{5}$$

where

$$p(x, k) = \sum_{n=0}^{\infty} p_n(x)\,k^{-n} \tag{6}$$

and $k$ is a large parameter, is [5]

$$y = e^{ik\varphi(x)}v(x, k) = e^{ik\varphi} \sum_{n=0}^{\infty} v_n(x)\,(ik)^{-n}. \tag{7}$$

Introducing (7) into (5) and equating coefficients of different powers of $k$ to zero gives

$$\varphi'(x)^2 = p_0(x), \tag{8}$$

$$2\varphi'(x)\, v_n'(x) + \left[\varphi''(x) + \sum_{r=0}^{\infty} p_{n+1-r}\right] v_n(x) = -\, v''_{n-1}(x); \tag{9}$$

$$n = 0, 1, \ldots; \qquad v_{-1} = 0.$$

The algorithm (7) leads to a complete set of formal asymptotic solutions of (5) in an $x$ region where the two roots of (8),

$$\varphi_{\pm}'(x) = \pm \sqrt{p_0(x)} \tag{10}$$

remain distinct. An isolated point where these roots coalesce, i.e., a zero of $p_0(x)$, is called a turning point. Near there the algorithm fails to yield analytic solutions [for analytic coefficients in (5)]. If the $p_n(x)$ have isolated singularities, the algorithm is also not valid, in general.

A connection between turning points and transition regions must be sought by comparing Eqs. (3) and (8) and examining the possibility of singularities in the coefficients of (1). Since $n^2(\mathbf{x}) > 0$ for all $\mathbf{x}$, no direct equivalent of the vanishing of $p_0(x)$ exists for (3). Further, in rectangular coordinates all coefficients in (1) are in fact regular everywhere including $n^2(x)$. The quadratic nature of $(\nabla\varphi)^2$ and $\phi'^2$ is the basis of our comparisons. Given an initial value problem for the eiconal equation, two distinct solutions exist if the data are noncharacteristic, due to the quadratic nature of $(\nabla\varphi)^2$. This corresponds to the case for (8), when $p_0(x)$ has no zeros, in the $x$ domain. For, just as the derivatives of $\phi_{\pm}(x)$ remain distinct everywhere, so do the gradients $\nabla\phi$ of both solutions to the eiconal equation, including the initial domain.

If the initial domain for (3) is the envelope of characteristics curves, i.e., a caustic, which we assume to be smooth, there are still two solutions for $\varphi$ above the caustic (as well as complex solutions below it). But on the caustic, the gradients $\nabla\varphi$ of both solutions coincide. Thus, the caustic is equivalent to a turning point for (5), in that at such a point both $\varphi_{\pm}'(x) = 0$, i.e., they coincide. We will see that it corresponds to a turning point of half-order (in Langer's terminology) [5], the correspondence being determined by the order of contact of the gradients of both solutions of (3). It is clear that to make a strict comparison between turning points and transition domains, we must require that the order of contact of the gradients $\nabla\varphi$ remain fixed on the initial domain. This rules out initial data which is noncharacteristic on part of the initial domain and characteristic elsewhere. Each part must be dealt with separately. The neighborhood of the transition points or curves in the initial domain where the data shifts from noncharacteristic to characteristic poses a special problem. For some cases, a perturbation method may be used to solve this case in terms of known transition region expansions. It will be discussed elsewhere.

If the data on the initial domain are characteristic, there are infinitely many solutions of (3) possible. Since $\nabla\varphi$ is now an interior derivative on the initial domain, it must also correspond to a turning point of some order for (5). In many diffraction problems there are generally two solutions of (3) which have a common set of characteristic curves, say along a shadow boundary. The order of contact of $\nabla\varphi$ for both solutions is of first order, and this corresponds to first order turning points for (5). If the contact of the two solutions is higher, we relate them to a higher order turning point. We again assume the data is characteristic everywhere on the initial domain.

Other types of initial value problems for (3) involve lower dimensional initial domains. Thus, for a point source, the initial domain is just the source point. There are two solutions to (3), the two halves of the characteristic conoid. The limiting gradients $\nabla\varphi$ of these solutions differ in direction at the initial point corresponding to incoming and outgoing waves. Thus, it corresponds to zero order turning point but some coefficients of (1) will become singular at that point in the appropriate coordinate system. If noncharacteristic initial data is given along a curve in a three dimensional problem, there are also two solutions, which may be considered as a single singular solution whose gradient is not defined along the curve. However, the limiting directions of the gradients $\nabla\varphi$ of both branches are distinct. Again, we have a zero order turning point situation but with a singularity in the coefficients of (1) along the curve. We will not discuss here transition regions which result from the intersection of two transition regions of the above types, e.g., a caustic with a cusp. Uniform expansions do exist for such regions and in some cases may be obtained by a perturbation method as indicated above. They will be discussed elsewhere as they correspond to exceptional cases in turning point theory which have not yet been treated in a general way.

We now investigate the solutions to the above initial value problems for the eiconal equation more closely. Expressing the eiconal equation in a local coordinate system with respect to the initial domain, such that one coordinate is along the normal direction, we may expand the solution in a series of powers (possibly fractional) of the normal coordinate. Such expansions were given by Keller [2] for a caustic and further analyzed by Ludwig [7]. The form of these expansions leads to the following results. The details have been carried out for $n^2 = 1$, but will not be presented here. If the data are noncharacteristic, we may write the two solutions of (3) as

$$\varphi_{\pm} = \theta \pm \omega, \tag{11}$$

where $\omega$ vanishes on the initial domain, while $\nabla\omega$ is not zero there. If $\theta = 0$, we have

$$\varphi_{\pm} = \pm \varphi, \tag{12}$$

which are the solutions for a point initial domain. Clearly, $\nabla\varphi$ is not zero at the initial point. If the initial domain is a smooth caustic, we may write

$$\varphi_{\pm} = \theta \pm \tfrac{2}{3}\beta^{3/2}, \tag{13}$$

where $\beta = 0$ on the caustic. For the characteristic case taking two solutions with first order contact on the initial domain, we have

$$\varphi_{\pm} = \theta \pm \tfrac{1}{2}\rho^2 \tag{14}$$

with $\rho = 0$ on the domain. If the contact of $\nabla\varphi_{\pm}$ is of higher order we need a higher power of $\rho$.

It is seen that, in fact, every solution has the form (11), with $\omega$ vanishing on the initial domain but not necessarily $\nabla\omega$. For noncharacteristic and point initial domains of the type considered above, $\nabla\omega \neq 0$ on the domain. For, we have

$$(\nabla\omega)^2 = \tfrac{1}{2}\{1 - \nabla\varphi_+ \cdot \nabla\varphi_-\}. \tag{15}$$

Thus, unless $\nabla\varphi_+$ and $\nabla\varphi_-$ have the same direction, in which case $\nabla\varphi_+ \cdot \nabla\varphi_- = 1$, $(\nabla\omega)^2$ will not vanish. On common characteristics or caustics, $\nabla\varphi_+ = \nabla\varphi_-$, but the level curves or surfaces $\varphi_{\pm} = \text{const}$ have contact of different order there. This is brought out by expressing $\omega$ in terms of $\beta$ and $\rho$, in (13) and (14), where $\nabla\beta$ and $\nabla\rho$ are nonzero on the initial domains. The order of vanishing of $\nabla\omega$ on the initial domain must be compared to the order to which $p_0(x)$ vanishes at the turning point. If $p_0(x) = 0(x)$, assuming the turning point to be at $x = 0$, it's of half-order, but for $p_0(x) = 0(x^2)$, the turning point is of first order. If $p_0(x)$ doesn't vanish at all, in some $x$ region, there is a zero order turning point. We note that the term $\theta$ which appears in (11), (13), and (14), results from the higher dimensionality of the initial domain in relation to the one dimensional problem for (5), where the turning point is zero dimensional. In fact, when the initial domain is zero dimensional as in (12), the two solutions of (3) differ only in sign, just as in (10). It is therefore, appropriate to compare the behavior of $\omega$ at the initial domain with that of $p_0(x)$ at the turning point. We shall refer to these initial domains as turning domains.

## 3. The Turning Domain Algorithm

The turning point algorithm used by Langer [5], Olver [6], and others, for (5) consists of the formal expansion

$$y(x, k) = A(x, k)\, w(z, k) + \frac{1}{k} B(x, k) \frac{dw(z, k)}{dz}, \tag{16}$$

where $A(x, k)$ and $B(x, k)$ are expanded in inverse powers of $k$ and $w(z, k)$ satisfies the "related equation"

$$\frac{d^2w}{dz^2} + k^2 q(z, k)\, w = 0.$$

The variable $z = z(x)$ is required to be a regular function of $x$. The related equation must be such that its solution is effected more easily than that of (5), while $q(z, k)$ depends on the nature of the turning point and the singularities of $p(x, k)$ in (5). The coefficients $A_n(x)$ and $B_n(x)$ in the expansions of $A$ and $B$ are required to be regular in the domain of interest, including the turning or singular point. That is, if (16) is introduced into (5) and the coefficients of $w$, $(dw/dz)$ and different powers of $k$ are equated to zero, the equations resulting for the $A_n$ and $B_n$ must have regular solutions. These conditions, together with the above on $z(x)$ lead to a complete determination of $q(z, k)$, which may itself be a power series in $k^{-1}$.

The discussion of Section 3 leads to the following turning region algorithm. Noting the presence of $\theta$ in both $\varphi_\pm$ and the relation between $\omega$ and $p_0(x)$, we look for solution of (1) in the form,

$$u = e^{ik\theta} W(\mathbf{x}, k). \tag{18}$$

The case $\theta = 0$ is also included in our discussion. Except for the case $\theta = 0$, we will assume from now on that $n^2(x) = 1$. This will lead to a simplification in our calculations but will not obscure the essential features of the uniform expansions as they exist for all choices of $n^2(\mathbf{x})$. The equation for $W(\mathbf{x}, k)$ is

$$\nabla^2 W + k^2 \left\{[1 - (\nabla\theta)^2]\, W + \frac{i}{k} [2\nabla\theta \cdot \nabla W + \nabla^2\theta W]\right\} = 0. \tag{19}$$

If we try to use the classical algorithm (2) for (19), that is,

$$W(\mathbf{x}, k) = e^{ik\sigma(\mathbf{x})} \sum_{n=0}^{\infty} w_n(\mathbf{x})\, (ik)^{-n}, \tag{20}$$

the new eiconal equation would be

$$(\nabla\sigma)^2 = 1 - (\nabla\theta)^2. \tag{21}$$

Noting (11)-(14), we can immediately identify the different types of turning domains for (19) from the order of vanishing of $(\nabla\sigma)^2$ on these domains.

The case of a lower dimensional turning domain may also be studied as follows. For a point domain with $\theta = 0$, in three and two dimensions, we introduce spherical and polar coordinates, respectively, with origin at that point into (1). Factoring out $r^{-1}$ and $r^{-1/2}$ in three and two dimensions, respectively, where $r$ is the radial coordinate, leads to a new equation whose coefficient of $u$ is $k^2n^2(x)$ and $k^2n^2(\mathbf{x}) + (1/4r^2)$ in three and two dimensions, respectively. The new equations still have zero order turning points at $r = 0$, but while the three dimensional case is nonsingular at that point, the two dimensional case is singular. The differentiated terms do have singularities, however. The special case of an axial caustic corresponds to a noncharacteristic lower dimensional initial value problem on an axis of symmetry, say the $z$ axis. Letting $r$ be a radial coordinate and factoring out $r^{-1/2}$ from (1), i.e., $u = r^{-1/2}V$ leads to

$$V_{rr} + V_{zz} + k^2\left[1 + \frac{1}{4k^2r^2}\right] V = 0, \tag{22}$$

assuming axial symmetry and $n^2 = 1$. This equation has a zero order turning domain, $r = 0$, but a singularity is there. This problem can also be studied directly from Eq. (1) without reduction to (22). This will be done below. Also, there is no need to factor out $r^{-1}$ and $r^{-1/2}$ in the point initial domain problem, to obtain uniform expansions. We have just done so to indicate the presence of a singularity at that point, in relation to the form of (5).

The turning region algorithm takes the form

$$u(\mathbf{x}, k) = e^{ik\theta}\left\{g(\mathbf{x}, k)\, H(\sigma, k) + \frac{1}{k}\, h(\mathbf{x}, k)\, \frac{dH(\sigma, k)}{d\sigma}\right\}, \tag{23}$$

where $g$ and $h$ are expanded in inverse powers of $k$, and $H(\sigma, k)$ satisfies the related equation

$$\frac{d^2H}{d\sigma^2} + k^2Q(\sigma, k)\, H = 0. \tag{24}$$

The bracketed item in (23) is actually the turning region algorithm for (19) and then (18) leads to the correct algorithm for (1). Suppose $\tau$ is a variable measuring distance normal to the turning domain. We then require that $\sigma$ be a regular function of $\tau$, just as $z(x)$ was a regular function of $x$ in the above. The coefficients $g_n$ and $h_n$ in the expansions of $g$ and $h$ must be regular at the turning domain and within the transition region. These conditions specify $Q(\sigma, k)$. While for Eq. (5) the turning point algorithm led to a complete set of asymptotic solutions, no such general result can be obtained for (1).

Rather, corresponding to each transition region, we obtain formal uniform asymptotic expansions, which must replace expansions of the form (2) in those regions. While the algorithm gives uniform expansions which reduce to expansions of the classical type (2) outside the transition regions, in most cases, except for source problems, the classical expansions are used to provide initial data for the transport equations for $g_n$ and $h_n$. In the shadow boundary problem, for instance, the initial data determines the form of $Q(\sigma, k)$, while for the caustic problem, $Q(\sigma, k)$ is independent of the data provided by the nonuniform expansions for $\varphi_\pm$.

We shall give related equations for all the above cases in the next section and consider some cases in detail, showing how to determine $Q(\sigma, k)$ and solve the transport equations for $g_n$ and $h_n$.

## 4. The Related Equations

The related equations for the cases of zero, half and first order turning points with no coincident singularities are [5]

$$\frac{d^2w}{dz^2} + k^2 w = 0, \tag{25}$$

$$\frac{d^2w}{dz^2} + k^2 z w = 0, \tag{26}$$

$$\frac{d^2w}{dz^2} + k^2 \left[z^2 + \frac{4i}{k}\,\hat{\alpha}(k)\right] w = 0, \tag{27}$$

respectively. $\hat{\alpha}(k)$ is a series in inverse powers of $k$ which depends on $p(\mathbf{x}, k)$ in (5). Related equations for (5), where $p_0(x)$ doesn't vanish and $p_2(x)$ behaves like $(c/x^2)$ at $x = 0$, have been given by Olver [6] in certain cases. They are Bessel's equations whose order depends on the constant $c$ in $p_2(x)$.

For Eq. (1), the zero order turning domain with no coincident singularities corresponds essentially to the case where the classical algorithm is valid. In this case, the related Eq. (24) has the form (25) so that no further discussion is needed. The half-order turning domain has the Airy Eq. (26) as its related equation. This case has been fully discussed by Ludwig [7]. The first order turning domain has not been discussed in full generality in the literature. The appropriate related equation has the form (27) which is a Weber equation. A uniform asymptotic expansion for the point source problem in two and three dimensions, i.e., a point initial domain, has been given by Babich [9]. The related equations for these cases with zero order turning points are Bessel's equations. For an axial caustic, i.e., a zero order turning region with a singularity, the related equation is Bessel's equation of zero order. This case has apparently not been discussed previously. We will consider the last three cases in some detail.

For a first order turning domain, we identify in the algorithm (23), $\sigma$ with $\rho$, as it was defined in (14). With this choice the condition on $\sigma$ in relation to $\tau$, given below Eq. (24), is satisfied since $\nabla\rho \neq 0$ on $\rho = 0$. As the related equation, we take

$$H''(k^{1/2}\rho) + k\left[\rho^2 - \frac{i\alpha}{k}\right] H(k^{1/2}\rho) = 0, \tag{28}$$

where $\alpha$ is a constant independent of $k$, which will be determined below. Then,

$$H''' + k\rho^2 H' = i\alpha H' - 2k^{1/2}\rho H. \tag{29}$$

We do not need the full expansion $\hat{\alpha}(k)$ to obtain regular functions $g_n$ and $h_n$ for our problem. We write (23) as

$$u(\mathbf{x}, k) = e^{ik\theta}\left\{g(\mathbf{x}, k)\, H(k^{1/2}\rho) + \frac{1}{ik^{1/2}}\, H'(k^{1/2}\rho)\, h(\mathbf{x}, k)\right\}, \tag{30}$$

where $h$ is slightly altered from (23). Then,

$$\begin{aligned} e^{-ik\theta}\{\nabla^2 u + k^2 u\} = {} & k^2[1 - (\nabla\theta)^2]\, Hg - ik^{3/2}[1 - (\nabla\theta)^2]\, H'h \\ & + ik^{3/2}(2\nabla\theta \cdot \nabla\rho)\, H'h + k(2\nabla\theta \cdot \nabla\rho)\, H''h + k(\nabla\rho)^2\, H''g \\ & - ik^{1/2}(\nabla\rho)^2\, H'''h + ik(2\nabla\theta \cdot \nabla g + \nabla^2\theta g)\, H \\ & + k^{1/2}(2\nabla\theta \cdot \nabla h + \nabla^2\theta h)\, H' + k^{1/2}[2\nabla\rho \cdot \nabla g + \nabla^2\rho g]\, H' \\ & - iH''[2\nabla\rho \cdot \nabla h + \nabla^2\rho h] + (\nabla^2 g)\, H + \frac{1}{ik^{1/2}}(\nabla^2 h)\, H' = 0 \end{aligned} \tag{31}$$

Now from $(\nabla\varphi_\pm)^2 = 1$ it follows that

$$(\nabla\theta)^2 + \rho^2(\nabla\rho)^2 = 1, \tag{32}$$

$$\nabla\theta \cdot \nabla\rho = 0, \tag{33}$$

but this can also be seen to be a consequence of (31) if (28) and (29) are used to simplify results. Conversely, if (32) and (33) are assumed, then the form of the related Eq. (28) for $H$ can be derived from the requirement that $g$ and $h$ must be regular at the turning domain. This will not be carried out here. Using (28), (29) and (32), (33), gives

$$\begin{aligned} & ikH\{2\nabla\theta \cdot \nabla g + \nabla^2\theta g + \alpha(\nabla\rho)^2 g + 2\rho(\nabla\rho)^2 h + 2\rho^2\nabla\rho \cdot \nabla h + \rho^2\nabla^2\rho h\} \\ & \quad + k^{1/2}H'\{2\nabla\rho \cdot \nabla g + \nabla^2\rho g + 2\nabla\theta \cdot \nabla h + \nabla^2\theta h + \alpha(\nabla\rho)^2 h\} \\ & \quad + H\{\nabla^2 g + 2\alpha\nabla\rho \cdot \nabla h + \alpha\nabla^2\rho h\} + \frac{1}{ik^{1/2}} H'[\nabla^2 h] = 0. \end{aligned} \tag{34}$$

We set

$$g(\mathbf{x}, k) = \sum_{n=0}^{\infty} g_n(\mathbf{x})\,(ik)^{-n}, \tag{35}$$

$$h(\mathbf{x}, k) = \sum_{n=0}^{\infty} h_n(\mathbf{x})\,(ik)^{-n}. \tag{36}$$

Equating coefficients of $H$, $H'$ and different powers of $k$ to zero results in,

$$\begin{aligned} &2\nabla\theta \cdot \nabla g_n + \nabla^2\theta g_n + \alpha(\nabla\rho)^2 g_n + \rho^2(2\nabla\rho \cdot \nabla h_n + \nabla^2\rho h_n) + 2\rho(\nabla\rho)^2 h_n \\ &= -\,[\nabla^2 g_{n-1} + \alpha(\nabla\rho \cdot \nabla h_{n-1} + \nabla^2\rho h_{n-1})]; \qquad n = 0, 1,\ldots; \quad g_{-1} = h_{-1} = 0. \end{aligned} \tag{37}$$

$$\begin{aligned} 2\nabla\theta \cdot \nabla h_n + \nabla^2\theta h_n + \alpha(\nabla\rho)^2 h_n + 2\nabla\rho \cdot \nabla g_n + \nabla^2\rho g_n = -\,\nabla^2 h_{n-1}; \\ n = 0, 1,\ldots; \qquad g_{-1} = h_{-1} = 0. \end{aligned} \tag{38}$$

Multiply (38) by $\rho$ and add and subtract it from (37). This gives, after some simplifications,

$$\begin{aligned} &2\nabla\varphi_\pm \cdot \nabla G_n^\pm + (\nabla^2\varphi_\pm + (\alpha \mp 1)\,(\nabla\rho)^2)\,G_n^\pm \\ &= -\,[\nabla^2 G_n^\pm + (\alpha \mp 1)\,(2\nabla\rho \cdot \nabla h_{n-1} + \nabla^2\rho h_{n-1})], \end{aligned} \tag{39}$$

where

$$G_n^\pm = g_n \pm \rho h_n\,, \tag{40}$$

and $\varphi_\pm$ is defined in (14).

To compare the transport Eqs. (39) with those given in (4), we set

$$G_n^\pm = \rho^{\lambda_\pm} Z_n^\pm \tag{41}$$

to obtain

$$\begin{aligned} &2\nabla\varphi_\pm \cdot \nabla Z_n^\pm + \nabla^2\varphi_\pm Z_n^\pm + (\alpha \mp 1 \pm 2\lambda_\pm)\,(\nabla\rho)^2\, Z_n^\pm \\ &= -\,\rho^{-\lambda_\pm}[\nabla^2 G_{n-1}^\pm + (\alpha \mp 1)\,(2\nabla\rho \cdot \nabla h_{n-1} + \nabla^2\rho h_{n-1})]. \end{aligned} \tag{42}$$

Letting

$$\lambda_\pm = \frac{1 \mp \alpha}{2} \tag{43}$$

makes the left hand side of (42) identical with that of (4). In fact, for $n = 0$, both equations are the same and for $\alpha = 1$, i.e., $\lambda_+ = 0$, the equations for $Z_n^+$ are identical with those in (4). Now, when $\rho = 0$, (39) and (42) are ordinary differential equations along that curve where $\nabla\varphi_+ = \nabla\varphi_-$, and according to our previous condition on the nature of the initial domain no other rays of $\varphi_+$ or $\varphi_-$can intersect the line $\rho = 0$. For such an intersection would imply that the ray system of $\varphi_+$ or $\varphi_-$ has a caustic on that line, so that two transition regions intersect there. In applying our expansion to a particular problem, we assume two classical expansions of the form (2) to be given outside the transition region, such that one expansion becomes singular at the characteristic turning line, while the other may or may not have a jump discontinuity across it. An example of such a case occurs near the totally reflected ray separating the domains of regular and total reflection resulting from a source above the interface [1]. There, the reflected field is not singular across the totally reflected characteristic line, but the diffracted or lateral wave becomes singular there. The choice $\alpha = 2$, makes both $G_n^{\pm}$ regular at turning line $\rho = 0$. Note that once $\lambda_+$ or $\lambda_-$ is chosen to make $G_n^+$ or $G_n^-$ regular, $\alpha$ is already determined and both exponents in $\rho^{\lambda\pm}$ are fixed.

In the case of diffraction by a thin screen [8], the shadow boundary plays the role of the turning domain. In this problem the classical expansion involving $\varphi_+$ (the incident wave), say, is given, and the diffracted wave expansion for $\varphi_-$ must be determined. The rays for $\varphi_-$ are given by geometrical diffraction theory and using the boundary conditions on the screen and the edge, a solution must be found. The edge is a lower dimensional initial domain for the solution of $(\nabla\varphi_-)^2 = 1$, so that $\varphi_-$ is singular along it. Thus, the transport equations for $\varphi_-$ are valid along rays which intersect at the edge, which itself intersects the shadow boundary. Yet, if we choose $\lambda_- = 1$ or $\alpha = 1$, the terms $G_n^-$ remains finite not only on the shadow boundary $\rho = 0$, away from the edge, but on the edge also. (See [8]). Then $\lambda_+ = 0$, and on evaluating the uniform expansion asymptotically, the expansion for $\varphi_+$ has a discontinuity across the shadow boundary as expected from geometrical optics. In general, we should pick $\alpha$ such that $G_0^{\pm}$ are regular functions near $\rho = 0$ and $G_0^+ = G_0^-$, when $\rho = 0$. Then, $h_0$ is also regular at $\rho = 0$ and all the $g_n$ and $h_n$ are regular.

For the case of an axial caustic, with $n^2 = 1$, we may use (30) rather than (1). The caustic curve, $r = 0$, is a zero order turning curve with a coincident singularity. The related equation is

$$H''(k, \omega) + \left[k^2 + \frac{1}{4\omega^2}\right] H(k\omega) = 0, \tag{44}$$

whose bounded solutions at $\omega = 0$ can be expressed as

$$H(k\omega) = (k\omega)^{1/2} J_0(k\omega), \tag{45}$$

where $J_0$ is the zero order Bessel function. Here, $\nabla\omega \neq 0$ on the caustic, and $(\omega/r)$ is finite at $r = 0$.

An alternative method for this problem involving (1) directly rather than (30) is based in the algorithm

$$u(\mathbf{x}, k) = e^{ik\theta}\left[J_0(k\omega)\, g(\mathbf{x}, k) + \frac{1}{ik}(k\omega)\, J_1(k\omega)\, h(\mathbf{x}, k)\right]. \tag{46}$$

Here, $\varphi_\pm = \theta \pm \omega$ as in (11), and $(\nabla\omega) \neq 0$ on the axis. The term $(k\omega)\, J_1(k\omega)$ is used in (46) rather than $J_0'(k\omega)$, to simplify the calculations, and use is made of the equations $J_0'(k\omega) = -J_1(k\omega)$ and $[(k\omega)\, J_1(k\omega)]' = (k\omega)\, J_0(k\omega)$. We insert (46) into (1), expand $g$ and $h$ in the usual way, and equate coefficients of $J_0$, $J_1$ and different powers of $k$ to zero. Adding and subtracting the coefficients of $J_0$ and $J_1$ for each power of $k$, we obtain

$$2\nabla\varphi_\pm \cdot \nabla G_n^\pm + \left[\nabla^2\varphi_\pm \pm \frac{(\nabla\omega)^2}{\omega}\right] G_n^\pm = -(\nabla^2 g_{n-1} \pm \omega\nabla^2 h_{n-1}), \tag{47}$$

where

$$G_n^\pm = g_n \pm \omega h_n, \tag{48}$$

and $\varphi_\pm$ is defined as above. The coefficient of $G_n^\pm$ is regular at $\omega = 0$. Using the assumed axial symmetry of the problem, so that the $g_n$ and $h_n$ are even functions of $\omega$, we conclude that all the $g_n$ and $h_n$ are regular at $\omega = 0$, in view of (47). Let

$$G_n^\pm = \omega^{1/2} Z_n^\pm, \tag{49}$$

then the equations for $Z_0^\pm$ are the usual transport equations of the type (4) for $n = 0$, while the higher order equations for $Z_n^\pm$, $n > 0$ have the same left side as (4). To determine the $G_n^\pm$, we note that for $\omega = 0$; we have $G_n^\pm = g_n$. Since an expansion of the classical type (2) is given for this problem involving either $\varphi_+$ or $\varphi_-$, $G_n^+$ or $G_n^-$, respectively, can be determined outside the transition region using the classical expansions' coefficients as data for (48). Then $g_n$ is known on the axial caustic $\omega = 0$, and this provides initial data for the second as yet undetermined set of transport terms $G_n^+$ or $G_n^-$.

The two dimensional point source problem can be considered as a special case of the axial caustic problem with $\theta = 0$. If we again assume $n^2 = 1$ as we did above, the solution of the related equation is itself an exact solution of (1). Therefore, we assume $n^2(x)$ is arbitrary but positive and regular. The strength of the source is given in the problem, and from this data a unique asymptotic solution must be determined. Equation (12) for $\varphi_\pm$ prevails for this problem, and we may use the algorithm (46) in the form,

$$u(\mathbf{x}, k) = \left\{ H_0^{(1)}(k\varphi)\, g(\mathbf{x}, k) + \frac{1}{ik}(k\varphi)\, H_1^{(1)}(k\varphi)\, h(\mathbf{x}, h) \right\}, \tag{50}$$

where we've set $\theta = 0$ and replaced $J_0$ by $H_0^{(1)}$, the zero order Hankel function of the first kind, $J_1$ by $H_1^{(1)}$ and $\omega$ by $\varphi$. The related equation is identical to that for the axial caustic, except that we now choose the singular solution, since this is a source problem. The term $g_0(\mathbf{x})$ in the expansion of $g(\mathbf{x}, k)$ is determined by the strength of the point source, while all other

$g_n$ and $h_n$ are required to be regular at $\varphi = 0$ and also to be such that all terms in (50) except the leading order term $g_0 H_0^{(1)}$ are finite at $\phi = 0$. Introducing (50) into (1) yields the transport equations identical with (47) except that $\varphi_\pm = \pm \phi$, so that we have, instead,

$$L(g_n) = 2\nabla\varphi \cdot \nabla g_n + \left[\nabla^2\varphi - \frac{n^2}{\varphi}\right] g_n = -\varphi\nabla^2 h_{n-1}, \tag{51}$$

$$L(\varphi h_n) = 2\nabla\phi \cdot \nabla(\phi h_n) + \left[\nabla^2\varphi - \frac{n^2}{\varphi}\right] (\varphi h_n) = -\nabla^2 g_{n-1}, \tag{52}$$

where we've written the equations of $g_n$ and $\varphi h_n$ separately by addition and subtraction of the single set in the form (47). Now the coefficients of $g_n$ and $\phi h_n$ in the above are regular and nonzero at $\varphi = 0$, so that the homogeneous forms of Eqs. (51) and (52) have regular nonzero solutions near $\varphi = 0$ for $g_n$ and $\phi h_n$. Note that if $\varphi h_n$ is nonzero at $\varphi = 0$, $h_n$ is singular there. Since $g_0 H_0{}^{(1)}$ is the only singular term permitted in expansion (50) at $\varphi = 0$, no nonzero solution of the homogeneous equation $L[g_n] = 0$ can be admitted for $n > 0$. Similarly, no nonzero solutions of the homogeneous equation $L(\varphi h_n) = 0$ can be admitted for all $n$. Then, if we insist that the particular solutions of (51) and (52) be regular at $\varphi = 0$, we obtain unique solutions for all equations in $g_n$, $n > 0$ and $h_n$, $n \geqslant 0$. The initial condition for $L(g_0) = 0$ is assumed to be given in the problem at $\phi = 0$. Therefore, (50) can be uniquely determined.

An alternative form of the source point expansion, in any number of dimensions greater than one, is given by Babich [9]. Specialized to two dimensions it can be written in our notation as

$$u(\mathbf{x}, k) = \sum_{n=0}^{\infty} (k\varphi)^n H_n^{(1)}(k\varphi) \frac{g_n(\mathbf{x})}{(ik)^{2n}}. \tag{53}$$

Using recursion formulas for Hankel functions, this form can reduce to the form (50). While Babich did not obtain (53) in this way, it can be obtained by applying to (1), Courant's [10] discussion of Hadamard's method for deriving the fundamental solution for second order hyperbolic and elliptic equations. This method applies to dimensions $N > 2$ as well, so we describe it generally. It proceeds from an expansion of the form

$$u(\mathbf{x}, k) = \sum_{n=0}^{\infty} S_n(\Gamma) g_n(\mathbf{x}), \tag{54}$$

where $\Gamma$ is defined as the solution of

$$(\nabla\Gamma)^2 = 4n^2(\mathbf{x})\, \Gamma \tag{55}$$

and

$$\frac{d}{d\Gamma} S_n(\Gamma) = S_{n-1}(\Gamma). \tag{56}$$

In our problem, $\Gamma = (\varphi_\pm)^2 = \varphi^2$. Using Hadamard's arguments at the point source [they are quite similar to those set forth in our discussion of the algorithm (50)], we would obtain

$$\Gamma S_{-2} + \frac{N}{2} S_{-1} = \Gamma S_0'' + \frac{N}{2} S_0' = 0 \tag{57}$$

as the equation for $S_0$, where $N$ is the dimension of the space. We note, however, that Hadamard's arguments remain valid in our problem if we replace (57) by

$$\Gamma S_{-2} + \frac{N}{2} S_{-1} + \frac{k^2}{4} S_0 = \Gamma S_0'' + \frac{N}{2} S_0' + \frac{k^2}{4} S_0 = 0. \tag{58}$$

We choose the singular solution of (57) and (58) and note that the addition of $(k^2/4)\, S_0$ in (58) does not enhance or diminish the singular behavior of $S_0$ at $\Gamma = 0$, but affects only its behavior as $\Gamma \to \infty$. Omitting this last term would render the expansion nonuniform. In this approach, we may consider (58) to be the related equation. Once $S_0$ is determined from (58), and for $N = 2$, we have

$$S_0(\Gamma) = H_0^{(1)}(k\sqrt{\Gamma}) = H_0^{(1)}(k\varphi). \tag{59}$$

the method of Courant to determine all the $g_n$ is valid, even though his choice of $S_0$ and, therefore, the $S_n$ for $n > 0$, differs. Without giving the details of the calculations, we state that the resulting expansion (54) has the form of (53).

Finally, we add that for a point source in three dimensions the procedure presented below Eq. (50) remains valid if $H_0{}^{(1)}$ is replaced by the singular solution of the appropriate related equation

$$\frac{d^2H}{d\varphi^2} + \frac{2}{\varphi}\frac{dH}{d\varphi} + k^2H = 0. \tag{60}$$

This equation leads to Hankel functions of half-integral order.

## 5. Conclusion

We have presented a general method for obtaining uniform asymptotic expansions for the reduced wave Eq. (1), valid in various transition regions. The requirement that the transition region or turning domain be of fixed type rules out many problems of interest, but it also enables a simple and useful comparison to be made with turning point theory for ordinary differential Eqs. (5). Turning domains, not of fixed type, have partial differential related equations, in general, rather than the ordinary differential equation encountered. In some cases, by introducing a suitable parameter, our expansions obtained above for fixed transition regions can be used for more general transition regions. These methods will be discussed elsewhere.

Further, the concept of "coincidence patterns" for root differences in turning point problems, introduced and developed by Langer, can be generalized to partial differential equations. Thus, uniform expansions for certain partial differential equations of higher order or systems of such equations can be generated by the knowledge of uniform expansions for (1) or a second order hyperbolic equation. This can occur if the generalized eiconal or characteristic equations possess coincidences of solutions similar to those considered above for the eiconal Eq. (3). Ludwig [7] has used this concept in some of his work.

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